

Quantum integrable systems : basic concepts and brief overview

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Abstract : An overview of the quantum integrable systems (QIS) is presented. Basic concepts of the theory are highlighted stressing on the unifying algebraic properties, which not only helps to generate systematically the representative Lax operators of different models, but also solves the related eigenvalue problem in an almost model independent way. Difference between the approaches in the integrable ultralocal and nonultralocal quantum models are explained and the interrelation between the QIS and other subjects are focused on

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1. Introduction

The theory and applications of nonlinear integrable systems is a vast subject with wide range of applications in diverse fields including biology, oceanography, atmospheric science, optics, plasma *etc.* The quantum aspect of the subject is a relatively new development. However the theory of quantum integrable systems (QIS) today has grown up into an enormously rich area with fascinating relations with variety of seemingly unrelated disciplines. The QIS in one hand is intimately connected with abstract mathematical objects like noncommutative Hopf algebra, braided algebra, universal R -matrix *etc.* and on the other hand is related to the concrete physical models in low dimensions including quantum spin chains, Hubbard model, Calogero-Sutherland model as well as QFT models like sine-Gordon (SG), nonlinear Schrödinger equation (NLS) *etc.* The deep linkage with the stat-mech problems, conformal field theory (CFT), knots and braids *etc.* is also a subject of immense importance.

In giving the account of this whole picture within this short span of time, I am really faced with the problem of Tristram Shendi [1], who in the attempt of writing his

autobiography needed two years for describing the rich experience of the first two days of his life and thus left us imagining when he would accomplish his mission. Therefore I will limit myself only to certain aspects of this important field and will be happy if it can arouse some of your interests in this fascinating subject.

We have to start possibly from an August day in 1834, when a British engineer historian. John Scott Russell had a chance encounter with a strange stable wave in the Union canal of Edinburgh [2]. Such *paradoxically* stable solutions will be observed again after many many years in the famous computer experiment of Fermi, Ulam and Pasta [3]. However only in the mid-sixties such fascinating phenomena will be understood fully as the solutions of nonlinear integrable systems and named as *Solitons* [4].

Formulation of the integrable theory of quantum systems started only in late seventies [7], though today many research groups all over the globe are engaged in active research in this field.

Mathematical basis of classical integrable systems was laid down mainly through the works of Sofia Kawalewskaya. Fuchs, Painlevé, Liouville and others [6]. There are many definitions of integrability; we however adopt the notion of integrability in the Liouville sense, where integrability means the existence of *action-angle* variables. That is, if in a Hamiltonian system $H[p(x, t), q(x, t)]$ given by the nonlinear equation

$$\dot{p} = -\frac{\delta H}{\delta q}, \quad \dot{q} = \frac{\delta H}{\delta p}, \quad (1.1)$$

it is possible to find a canonical transformation $(p(x, t), q(x, t)) \rightarrow (a(\lambda), b(\lambda, t))$, such that the new Hamiltonian becomes dependent only on the action variables, i.e. $H = H[a(\lambda)]$, then the system may be called *completely integrable*. In this case the dynamical equations :

$$\dot{a} = -\frac{\delta H}{\delta b} = 0, \quad \dot{b} = \frac{\delta H}{\delta a} = \omega, \quad \text{can be trivially solved and moreover we get } a(\lambda) \text{ as the}$$

generator of the conserved quantities. The number of such independent set of conserved quantities in integrable systems coincides with the degree of freedom of the system and in field models it becomes infinite. One of these conserved quantities may be considered as the Hamiltonian. The inverse scattering method (ISM) [5] is an effective method for solving nonlinear equations. The important feature of ISM is that, instead of attacking the nonlinear equation (1.1) directly, it constructs the corresponding linear scattering problem

$$T_x(x, \lambda) = L(q(x, t), p(x, t), \lambda) T(x, \lambda), \quad (1.2)$$

where the Lax operator $L(q, p, \lambda)$ depending on the fields q, p and the *spectral parameter* λ contains all information about the original nonlinear system and may serve therefore as the representative of a concrete model. The field q in ISM acts as the scattering potential. The aim of ISM is to find precisely the canonical mapping from the action-angle variables to the original field and using it to construct the exact solutions for the original nonlinear equation. Soliton is a special solution, which corresponds to the reflectionless ($b(\lambda) = 0$) potential.

2. Examples of integrable systems

Let us see some concrete examples of the Lax operators associated with well known models to get an idea about the structure of this immensely important object in the integrable systems.

I. Trigonometric Class :

1. Sine-Gordon (SG) model (Equation and Lax operator)

$$u(x, t)_{tt} - u(x, t)_{xx} = \frac{m^2}{\eta} \sin(\eta u(x, t)),$$

$$\mathcal{L}_{SG} = \begin{pmatrix} ip, & m \sin(\lambda - \eta u) \\ m \sin(\lambda + \eta u), & -ip \end{pmatrix}, p = u. \quad (2.1)$$

2. Liouville model (LM) (Equation and Lax operator)

$$u(x, t)_{tt} - u(x, t)_{xx} = \frac{1}{2} e^{2\eta u(x, t)}, \quad \mathcal{L}_{LM} = i \begin{pmatrix} p, & \xi e^{\eta u} \\ \frac{1}{\xi} e^{-\eta u}, & -p \end{pmatrix}. \quad (2.2)$$

3. Anisotropic XXZ spin chain (Hamiltonian and Lax operator)

$$\mathcal{H} = \sum_n \left(\sigma_n^1 \sigma_{n+1}^1 + \sigma_n^2 \sigma_{n+1}^2 + \cos \eta \sigma_n^3 \sigma_{n+1}^3 \right),$$

$$L_n(\xi) = \begin{pmatrix} \sin(\lambda + \eta \sigma_n^3), & 2i \sin \alpha \sigma_n^- \\ 2i \sin \eta \sigma_n^+, & \sin(\lambda - \eta \sigma_n^3) \end{pmatrix}. \quad (2.3)$$

II. Rational Class :

1. Nonlinear Schrödinger equation (NLS) (Hamiltonian and Lax operator)

$$i\psi(x, t)_t + \psi(x, t)_{xx} + \eta(\psi^\dagger(x, t)\psi(x, t))\psi(x, t) = 0,$$

$$\mathcal{L}_{NLS}(\lambda) = \begin{pmatrix} \lambda, & \eta^{\frac{1}{2}} \psi \\ \eta^{\frac{1}{2}} \psi^\dagger, & -\lambda \end{pmatrix}. \quad (2.4)$$

2. Toda chain (TC) (Hamiltonian and Lax operator)

$$H = \sum_i \left(\frac{1}{2} p_i^2 + e^{(q_i - q_{i+1})} \right), \quad L_n(\lambda) = \begin{pmatrix} p_n - \lambda & e^{q_n} \\ -e^{-q_n} & 0 \end{pmatrix}. \quad (2.5)$$

Let us note the following important points on the structure of the above Lax operators.

- (i) The Lax operator description generalises also to the quantum case [7,8]. Its elements depend, apart from the spectral parameter λ , also on the field operators u, p or ψ, ψ^\dagger etc and therefore the quantum $L(\lambda)$ -operators are unusual matrices with noncommuting matrix elements. This intriguing feature leads to nontrivial underlying algebraic structures in QIS.

- (ii) The off-diagonal elements (as ψ, ψ^\dagger in (2,4) and σ^-, σ^+ in (2,3)) involve *creation* and *annihilation* operators while the diagonal terms are the number like operators. It is obvious that under matrix multiplication also this property is maintained, which has important implications, as we will see below.
- (iii) The first three models, though diverse looking, belong to the same trigonometric class. Similarly the rest of the models represents the rational class. The fact signals about a fascinating universal behaviour in integrable systems based on its rich algebraic structure.

3. Notion of quantum integrability

Note that the Lax operators are defined locally at a point x , or if we discretise the space, at every lattice point i . However, since the integrability is related to the conserved quantities, which are indeed global objects, we also have to define some global entries out of the local description of the Lax operators. Such an object can be formed by matrix multiplying Lax operators at all points as

$$T(\lambda) = \prod_{i=1}^N L_i(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \quad (3.1)$$

Here the global operators $B(\lambda), C(\lambda)$ are related to the angle like variables, while $A(\lambda), D(\lambda)$ are like action variables and $\tau(\lambda) = \text{tr} T(\lambda) = A(\lambda) + D(\lambda)$ generates the conserved operators: $\ln \tau(\lambda) = \sum_j C_j \lambda^j$. For ensuring integrability one must show for the conserved quantities that $[H, C_m] = 0, [C_n, C_m] = 0$, which is achieved by a key requirement on the Lax operators (for a large class of models) given by the matrix relation known as the Quantum Yang-Baxter equation (QYBE)

$$R_{12}(\lambda, \mu) L_{1i}(\lambda) L_{2i}(\mu) = L_{2i}(\mu) L_{1i}(\lambda) R_{12}(\lambda, \mu), \quad (3.2)$$

with the appearance of a 4×4 -matrix $R(\lambda, \mu)$ with c-number functions of spectral parameters, satisfying in turn the YBE

$$R_{12}(\lambda, \mu) R_{13}(\lambda, \gamma) R_{23}(\mu, \gamma) = R_{23}(\mu, \gamma) R_{13}(\lambda, \gamma) R_{12}(\lambda, \mu). \quad (3.3)$$

Due to some deep algebraic property related to the Hopf algebra the same QYBE also holds globally:

$$R_{12}(\lambda, \mu) T_1(\lambda) T_2(\mu) = T_2(\mu) T_1(\lambda) R_{12}(\lambda, \mu), \quad (3.4)$$

with the notations $T_1 = T \otimes I, T_2 = I \otimes T$. Taking the trace of relation (3.4), (since under the trace R -matrices can rotate cyclically and thus cancel out) one gets $[\tau(\lambda), \tau(\mu)] = 0$, establishing the commutativity of C_n for different n 's and hence proving the quantum integrability.

The QYBE (3.4) represents in the matrix form a set of commutation relations between *action* and *angle* variables, which can be obtained by inserting in (3.4)

matrix (3.1) for T and the solution for quantum $R(\lambda, \mu)$ -matrix, which may be given by

$$R(\lambda) = \begin{pmatrix} f(\lambda) & & & \\ & 1 & f_1 & \\ & f_1 & 1 & \\ & & & f(\lambda) \end{pmatrix}. \quad (3.5)$$

The solutions are usually of only two different types (we shall not speak here of more general elliptic solutions), *trigonometric* with

$$f = \frac{\sin(\lambda + \eta)}{\sin \lambda}, \quad f_1 = \frac{\sin \eta}{\sin \lambda} \quad (3.6)$$

and the *rational* with

$$f = \frac{\lambda + \eta}{\lambda}, \quad f_1 = \frac{\eta}{\lambda}. \quad (3.7)$$

4. Exact solution of eigenvalue problem through algebraic Bethe ansatz

Such generalised commutation relations dictated by the QYBE are of the form

$$A(\lambda)B(\mu) = f(\mu - \lambda)B(\mu)A(\lambda) + \dots, \quad (4.1)$$

$$D(\lambda)B(\mu) = f(\lambda - \mu)B(\mu)D(\lambda) + \dots, \quad (4.2)$$

together with the trivial commutations for $[A(\lambda), A(\mu)] = [B(\lambda), B(\mu)] = [D(\lambda), D(\mu)] = [A(\lambda), D(\mu)] = 0$ etc.

It is now important to note that the *off* diagonal element $B(\lambda)$ acts like an creation operator (induced by the local creation operators of $L(\lambda)$ as argued above). Therefore if one can solve the quantum eigenvalue problem

$$H|m\rangle = E_m |m\rangle \quad (4.3)$$

or more generally

$$\tau(\lambda)|m\rangle = \Lambda_m(\lambda)|m\rangle \quad (4.4)$$

the eigenvalue problem for all C_n 's can be obtained simultaneously by simply expanding $\Lambda(\lambda)$ as

$$C_1|m\rangle = \Lambda'_m(0)\Lambda_m^{-1}(0)|m\rangle, \quad C_2|m\rangle = \left(\Lambda'_m(0)\Lambda_m^{-1}(0)\right)'|m\rangle \quad (4.5)$$

etc. The m -particle state $|m\rangle$ may be considered to be created by $B(\lambda_i)$ acting m times on the pseudovacuum $|0\rangle$:

$$|m\rangle = B(\lambda_1)B(\lambda_2)\dots B(\lambda_m)|0\rangle. \quad (4.6)$$

Therefore for solving (4.4) through the Bethe ansatz, we have to drag $\alpha(\lambda) = A(\lambda) + D(\lambda)$ through the string of $B(\lambda_j)$'s without spoiling their structures (and thereby preserving the eigenvector) and hit finally the pseudovacuum giving $A(\lambda) |0\rangle = \alpha(\lambda) |0\rangle$ and $D(\lambda) |0\rangle = \beta(\lambda) |0\rangle$. Notice that for this purpose (4.1,4.2) coming from the QYBE are the right kind of relations. (the other type of unwanted terms are usually present in the LHS in lattice models ((as ... in (4.1,4.2)), which however may be removed by the Bethe equations for determining the parameters λ_j , induced by the periodic boundary condition. In case of field models such terms are absent and λ_j become arbitrary.) As a result we finally solve the eigenvalue problem to yield

$$\Lambda_m(\lambda) = \prod_{j=1}^m f(\lambda_j - \lambda) \alpha(\lambda) + \prod_{j=1}^m f(\lambda - \lambda_j) \beta(\lambda). \quad (4.7)$$

5. Universality in integrable systems

The structure of the eigenvalue $\Lambda_m(\lambda)$ reveals the curious fact that apart from the $\alpha(\lambda)$, $\beta(\lambda)$ factors it depends basically on the nature of the function $f(\lambda - \lambda_j)$, which are known trigonometric or rational functions given by (3.6) or (3.7) and thus is the same for all models belonging to the same class. Model dependence is reflected only in the form of $\alpha(\lambda)$ and $\beta(\lambda)$ factors. Therefore the models like SG, Liouville and XXZ chain belonging to the trigonometric class share similar type of eigenvalue relations (with specific forms for $\alpha(\lambda)$ and $\beta(\lambda)$). This deep rooted universality feature in integrable systems carries important consequences.

5.1. Generation of models :

One may start with the trigonometric solution (3.6) for the R -matrix and consider a generalised model with Lax operator

$$L_l(\lambda) = \begin{pmatrix} \sin(\lambda + \eta s^3), & \sin \eta S^- \\ \sin \eta S^+, & \sin(\lambda - \eta s^3) \end{pmatrix} \quad (5.1)$$

with the abstract operators s^3, S^\pm belonging to the quantum algebra (QA) $U_q(su(2))$:

$$[S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = [2S^3]_q. \quad (5.2)$$

where $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sin(\alpha x)}{\sin \alpha}$, $q = e^{i\alpha}$. Following the above Bethe ansatz procedure

the eigenvalue would naturally be like (4.7) and different realisation of the quantum algebra (5.2) would derive easily the eigenvalues for concrete models belonging to this class. At the same time the Lax operators of these models can also be generated from (5.1) in a systematic way.

For example,

$$S^\pm = \frac{1}{2} \sigma^\pm, \quad s^3 = \frac{1}{2} \sigma^3 \quad (5.3)$$

constructs from (5.1) the Lax operator of the spin- $\frac{1}{2}$ XXZ-chain and describes the Bethe-ansatz solution for the suitable choice of $\alpha(\lambda)$ and $\beta(\lambda)$. Similarly,

$$S_n^3 = u_n, \quad S_n^- = g(u_n) e^{i\Delta} p_n, \quad S_n^+ = (S_n^-)^\dagger, \quad (5.4)$$

with $g(u_n) = \left[1 + \frac{1}{2} m^2 \Delta^2 \cos 2\eta \left(u_n + \frac{1}{2} \right) \right]^{\frac{1}{2}}$ yields (lattice) sine-Gordon model. At $\Delta \rightarrow 0$ one gets the SG field model with the Lax operator obtained as $L_n = I + \Delta \mathcal{L}(x) + O(\Delta)$.

All the conserved quantities of the model including the Hamiltonian can in principle be derived using the Lax operator. In fact a more general form of the ancestor Lax operator than that of (5.1) exists corresponding to the same trigonometric R -matrix, the explicit form of which can be found in ref. [16]. Concrete realisations of such ancestor models generates various quantum integrable models (in addition to those already mentioned) like quantum Derivative NLS, Ablowitz-Ladik model, relativistic Toda chain *etc.* The Bethe ansatz solutions for these models also can be obtained (with specific case-dependent difficulties) following the scheme for their ancestor model, which as mentioned above is almost model independent and same for all models of the same class.

At $q \rightarrow 1$ limit, $R_{\text{alg}} \rightarrow R_{\text{rat}}$ and given by the elements (3.7). The ancestor model also reduces to the corresponding rational form

$$L_\tau(\lambda) = \begin{pmatrix} \lambda + \eta s^3 & \eta s^- \\ \eta s^+ & \lambda - \eta s^3 \end{pmatrix}. \quad (5.5)$$

The underlying QA (5.2) becomes the standard $su(2)$ algebra

$$[s^3, s^\pm] = \pm s^\pm, \quad [s^+, s^-] = 2s^3. \quad (5.6)$$

Such rational ancestor model (or with more general form [16]) in its turn reduces also to quantum integrable models like spin- $\frac{1}{2}$ XXX chain, NLS model, Toda chain *etc.* For example, spin- $\frac{1}{2}$ representation $s^a = \frac{1}{2} \sigma^a$ gives the Lax operator of XXX chain from (5.5), while the mapping from spin to bosonic operators given by Holstein-Primakov transformation

$$s^3 = s - \Delta \psi^\dagger \psi, \quad s^- = \Delta^{\frac{1}{2}} (2s - \Delta \psi^\dagger \psi)^{\frac{1}{2}} \psi^+, \quad s^+ = (s^-)^\dagger \quad (5.7)$$

leads to the quantum integrable Lattice NLS model. Similarly, the Toda chain can be derived from the ancestor model of [16]. The Bethe ansatz solutions for these descendant models mimics also the scheme for their ancestor model with rational R -matrix.

Thus for both the trigonometric and rational classes one can construct the Lax operators and solve the eigenvalue problem exactly through Bethe ansatz in a systematic way. This unifies diverse models of the same class as descendants from the same ancestor model and at the same time realisations like (5.4) gives a criterion for defining *integrable nonlinearity* as different nonlinear realisations of the underlying QA. This fact of the close relationship between seemingly diverse models also explains in a way the *strange* statements often met in other contexts like 'Quantum Liouville model is equivalent to spin $(-\frac{1}{2})$ anisotropic chain' [31] or 'High energy scattering of hadrons in QCD is described by the Heisenberg model with noncompact group' [30].

5.2. Algebraic structure of integrable systems :

The underlying QA, as mentioned before, exhibits Hopf algebra property. The most prominent characteristic of it is the coproduct structure given by

$$\Delta(s^3) = s^3 \otimes I + I \otimes s^3, \quad \Delta(S^\pm) = S^\pm \otimes q^{s^3} + q^{-s^3} \otimes S^\pm, \quad (5.8)$$

This means that if $S_1^\pm = S^\pm \otimes I$ and $S_2^\pm = I \otimes S^\pm$ satisfy the QA separately, then their tensor product $\Delta(S^\pm)$ given by (5.8) also satisfies the same algebra. This Hopf algebraic property of the QA induces the crucial transition from the local QYBE (3.2) to its tensor product given by the global equation (3.4), which in turn guarantees the quantum integrability of the system as shown above.

The QIS described above are known as the *ultralocal* models. They are the standard and the most studied ones. The ultralocality refers to their common property that the Lax operators of all such models at different lattice points $i \neq j$ commute : $[L_{1i}, L_{2j}]$. Note that this is consistent with the property : $[S_i^a, S_j^b] = 0$ for the generators of the quantum algebra described above. This ultralocality is actively used for transition from the local to the global QYBE, *i.e.* in establishing their quantum integrability.

Note that the standard matrix multiplication rule

$$(A \otimes B)(C \otimes D) = (AC \otimes BD) \quad (5.9)$$

which holds due to the commutativity of $B_2 = I \otimes B$ and $C_1 = C \otimes I$, remains also valid for the ultralocal Lax operators with the choice

$$A = L_{i+1}(\lambda), \quad B = L_{i+1}(\mu), \quad C = L_i(\lambda), \quad D = L_i(\mu). \quad (5.10)$$

Therefore starting from the local QYBE (3.2) at $i + 1$ point, multiplying with the same relation at i and subsequently using (5.9) with (5.10) one globalises the QYBE and repeating the step for N times obtains finally the global QYBE (3.4). This in turn leads to the commuting traces $\tau(\lambda) = \text{Tr}T(\lambda)$ giving commuting conserved quantities $C_n, n = 1, 2 \dots N$.

6. Nonultralocal models and braided extension of QYBE

However, there exists another class of models, known as nonultralocal models (NM) with the property $[L_{1i}, L_{2j}] \neq 0$, for which the trivial multiplication property (5.9) of quantum algebra fails and it needs generalisation to the braided algebra [9], where the noncommutativity of B_2, C_1 could be taken into account. Consequently the QYBE should be generalised for such models. Though many celebrated models, e.g. quantum KdV model. Supersymmetric models, nonlinear σ models, WZWN etc. belong to this class, apart from few [10,11] not enough studies have been devoted to this problem. The generalised QYBE for nonultralocal systems with the inclusion of braiding matrices Z (nearest neighbour braiding) and \tilde{Z} (nonnearest neighbour braiding) may be given by

$$\begin{aligned} R_{12}(u-v)Z_{21}^{-1}(u, v)L_{1j}(u)\tilde{Z}_{21}(u, v)L_{2j}(v) \\ = Z_{12}^{-1}(v, u)L_{2j}(v)\tilde{Z}_{12}(v, u)L_{1j}(u)R_{12}(u-v). \end{aligned} \quad (6.1)$$

In addition, this must be complemented by the braiding relations

$$\begin{aligned} L_{2j+1}(v)Z_{21}^{-1}(u, v)L_{1j}(u) = \tilde{Z}_{21}^{-1}(u, v)L_{1j}(u)\tilde{Z}_{21}(u, v) \\ \times L_{2j+1}(v)\tilde{Z}_{21}^{-1}(u, v) \end{aligned} \quad (6.2)$$

at nearest neighbour points and

$$\begin{aligned} L_{2k}(v)\tilde{Z}_{21}^{-1}(u, v)L_{1j}(u) = \tilde{Z}_{21}^{-1}(u, v)L_{1j}(u)\tilde{Z}_{21}(u, v) \\ \times L_{2k}(v)\tilde{Z}_{21}^{-1}(u, v) \end{aligned} \quad (6.3)$$

with $k > j + 1$ answering for the nonnearest neighbours. Note that along with the usual quantum $R_{12}(u-v)$ -matrix like (3.5) additional \tilde{Z}_{12}, Z_{12} matrices appear, which can be (in-)dependent of the spectral parameters and satisfy a system of Yang-Baxter type relations [11]. Due to appearance of Z matrices however one faces initial difficulty in trace factorisation unlike the ultralocal models. Nevertheless, in most cases one can bypass this problem by introducing a $K(u)$ matrix and defining $t(u) = \text{tr}(K(u)\mathcal{T}(u))$ as commuting matrices [20,11] for establishing the quantum integrability for nonultralocal models. Though a wellframed theory for such systems is yet to be achieved one can derive the basic equations for a series of nonultralocal models in a rather systematic way from the general relations (6.1–6.3) by particular explicit choices of Z, \tilde{Z} and R -matrices [11,19]. The models which can be covered through this scheme are

- (i) *Nonabelian Toda chain* [12]

$$\tilde{Z} = 1, Z = I + i\hbar(e_{22} \otimes e_{12}) \otimes \pi.$$

- (ii) *Current algebra in WZWN model* [29]

$$\tilde{Z} = 1 \text{ and } Z_{12} = R_{q12}^{-1}, \text{ where } R_q^\pm \text{ is the } \lambda \rightarrow \pm\infty \text{ limit of the trigonometric } R(\lambda)\text{-matrix.}$$

(iii) *Coulomb gas picture of CFT* [14]

$$\tilde{Z} = 1 \text{ and } Z_{12} = q^{-\sum H_i \otimes H_i}.$$

(iv) *Nonultralocal quantum mapping* [13]

$$\tilde{Z} = 1 \text{ and } Z_{12}(u_2) = 1 + \frac{h}{u_2} \sum_{\alpha}^{N-1} e_{N\alpha} \otimes e_{\alpha N}.$$

(v) *Integrable model on moduli space* [15]

$$\tilde{Z} = Z_{12} = R_q^{-}.$$

(vi) *Supersymmetric models*

$$Z = \tilde{Z} = \sum \eta_{\alpha\beta} g_{\alpha\beta} \text{ where } \eta_{\alpha\beta} = e_{\alpha\alpha} \otimes e_{\beta\beta} \text{ and } g = (-1)^{\tilde{\alpha}\tilde{\beta}} \text{ with supersymmetric grading } \tilde{\alpha}.$$

(vii) *Anyonic type SUSY model*

$$Z = \tilde{Z} = \sum \eta_{\alpha\beta} \tilde{g}_{\alpha\beta} \text{ with } \tilde{g}_{\alpha\beta} = e^{i\theta\tilde{\alpha}\tilde{\beta}}$$

(viii) *Quantum mKdV model* [18]

$$\tilde{Z} = 1, Z_{12} = Z_{21} = q^{-\frac{1}{2}\sigma^3 \otimes \sigma^3} \text{ and the trigonometric } R(u) \text{ matrix}$$

(ix) *Kundu-Eckhaus equation* [17]

Classically integrable NLS equation with 5th power nonlinearity

$$i\psi_t + \psi_{xx} + \kappa(\psi^\dagger \psi)\psi + \theta^2(\psi^\dagger \psi)^2 \psi + 2i\theta(\psi^\dagger \psi)_x \psi = 0, \quad (6.4)$$

as a quantum model involves anyonic type fields : $\psi_n \psi_m = e^{i\theta} \psi_m \psi_n$, $n > m$: $[\psi_m, \psi_n^\dagger] = 1$. The choice $\tilde{Z} = 1, Z = \text{diag}(e^{i\theta}, 1, 1, e^{i\theta})$ and the rational R matrix constructs the braided QYBE. The trace factorisation problem has not been solved.

Other models of nonultralocal class are the wellknown *Calogero-Sutherland* (CS) and *Haldae-Shastry* (HS) models with interesting long-range interactions. Spin extension of the CS model may be given by the Hamiltonian [34,32]

$$H_{cs} = \sum_{j=1}^N p_j^2 + 2 \sum_{1 \leq j < k \leq N} (a^2 - aP_{jk}) V(x_j - x_k) \quad (6.5)$$

with $[x_j, p_k] = i\delta_{jk}$, where the potential $V(x_j - x_k) = \frac{1}{(x_j - x_k)^2}$ for nonperiodic and $\frac{1}{\sin^2(x_j - x_k)}$ for the periodic model. P_{jk} is the permutation operator responsible for exchanging the spin states of the j -th and the k -th particles. In the absence of the operator P_{jk} , (6.5) turns into the original CS model without spin.

The spin CS model exhibits many fascinating features, namely its conserved quantities including the Hamiltonian exhibit Yangian symmetry, the eigenvalue problem can be solved exactly using Dunkl operators, the ground state is a solution of the Knizhnik-Zamolodchikov equation, the system can be viewed as the free anyonic gas related to the notion of fractional statistics *etc.* [32]. Though the satisfactory formulation of quantum integrability of the model by braided QYBE has not yet been achieved, this was done through an alternative procedure using operators L and M and showing $[H_{cs}, L] = [L, M]$ [33].

Remarkably, at $a \rightarrow \infty$ the Hamiltonian of the CS model (6.5) for the periodic case reduces to the HS model [35]

$$H_{hs} = \sum_{j < k} \frac{P_{jk}}{\sin^2(x_j - x_k)}. \quad (6.6)$$

This discretized long-range interacting spin chain like model seems to be less well understood and its Lax operator description difficult to find.

7. Interrelation of QIS with other fields

As I have mentioned in the introduction the quantum integrable system is intimately connected with various other branches of physics and mathematics. Therefore the knowledge and techniques of QIS is often helpful in understanding and solving other problems. Here we briefly touch upon some of these relations just to demonstrate the wide range of applicability of the theory of integrable systems.

7.1. Relation with statistical systems :

The $(1+1)$ dimensional quantum systems are linked with 2 dimensional classical statistical systems and the notion of integrability is equivalent in both these cases. For integrable statistical systems the QYBE (3.2) and the YBE (3.3) becomes the same and leads similarly to the commuting transfer matrix $\tau(\lambda)$ for different λ .

Let us examine a classical statistical model known as vertex model by considering 2-dimensional array of $N \times M$ lattice points connected by the bonds assigned with +ve (-ve) signs or equivalently, with right, up (left, down) arrows in a random way. The partition function Z of this system may be given starting from local properties, *i.e.* by finding the probability of occurrence of a particular configuration at a fixed lattice point i . For two allowed signs on each bond, $4 \times 4 = 16$ possible arrangements arises at each lattice point. Setting the corresponding Boltzmann weights $\omega_j = e^{-\alpha_j \beta}$ as the matrix elements of a 4×4 -matrix, we get the $R_{12}^{(i)}$ -matrix with crucial dependence on spectral parameter λ . The configuration probability for a string of N -lattice sites in a row may be given by the transfer matrix $\tau(\alpha, \beta) = \text{tr}(\prod_i^N R^{(i)})$. For calculating the partition function involving M such strings, one has to repeat the procedure M times to give $Z = \text{tr}(\prod^M T) = \text{tr}(T^M)$.

The YBE (3.3) restricts the solution of the R -matrix to integrable models. However the R matrix with 16 different Boltzmann weights, representing in general a 16-vertex model is difficult to solve. Therefore we impose some extra symmetry and conditions on the R -matrix by requiring the charge conserving symmetry $R_{kl}^{ij} \neq 0$, only when $k + l = j + i$, along with a charge or arrow reversing symmetry (see Figure 1).

$$R_{--}^{-} \left(\begin{array}{c} \uparrow \\ \leftarrow \rightarrow \\ \downarrow \end{array} \right) = R_{--}^{-} \left(\begin{array}{c} \uparrow \\ \leftarrow \rightarrow \\ \downarrow \end{array} \right) \equiv a(\lambda), \quad R_{--}^{-} \left(\begin{array}{c} \uparrow \\ \leftarrow \rightarrow \\ \downarrow \end{array} \right) = R_{--}^{+} \left(\begin{array}{c} \uparrow \\ \leftarrow \rightarrow \\ \downarrow \end{array} \right) \equiv b(\lambda),$$

$$R_{+-}^{+} \left(\begin{array}{c} \uparrow \\ \leftarrow \rightarrow \\ \downarrow \end{array} \right) = R_{+-}^{+} \left(\begin{array}{c} \uparrow \\ \leftarrow \rightarrow \\ \downarrow \end{array} \right) \equiv c(\lambda).$$

Figure 1. Boltzmann weights of the 6-vertex model constituting the elements of the R -matrix.

Using an overall normalisation it leads to a 6-vertex model for which the R -matrix is given exactly by (3.5), which in turn represents the Lax operator of the XXZ spin- $\frac{1}{2}$ chain as constructed above. Thus we see immediately the similarity between statistical and the quantum systems in their construction of R -matrices, transfer matrix, integrability equations *etc.* This deep analogy goes also through all the steps in solving the eigenvalue problem by Bethe ansatz in both the systems, *e.g.* vertex models in integrable statistical systems and the spin chains in QIS [21,22].

7.2. Interrelation between QIS and CFT :

There exists deep interrelation between these two two-dimensional systems, first revealed perhaps by Zamolodchikov [24] by showing that, if CFT is perturbed through relevant perturbation and the system goes away from criticality it might generate hierarchies of integrable systems. For example, $c = \frac{1}{2}$ CFT perturbed by the field $\sigma = \phi_{(1,2)}$ as $H = H_{1/2} + \hbar \sigma \int \sigma(x) d^2 x$, represents in fact the Ising model at $T = T_c$ with nonvanishing magnetic field h . Similarly the WZWN model perturbed by the operator $\phi_{(1,3)}$ generates integrable restricted sine-Gordon (RSG) model. Under such perturbations the trace of the stress tensor, unlike pure CFT, becomes nonvanishing and generates in principle infinite series of integrals of motion associated with the integrable systems.

In recent years, this relationship has also been explored by stretching it in a sense from the opposite direction. The aim was to describe CFT through *massless* S -matrix [23] starting from the theory of integrable systems. This alternative approach based on the quantum KdV model attempts to capture the integrable structure of CFT. Note that the conformal symmetry of CFT is generated by its energy-momentum tensor $T(u) = -\frac{c}{24} + \sum_{n=-\infty}^{\infty} L_{-n} e^{inu}$, with L_n satisfying the Virasoro algebra. The operators

$I_{2k-1} = \frac{1}{2\pi} \int_0^{2\pi} du T_{2k}(u)$, with $T_{2k}(u)$, depending on various powers and derivatives of $T(u)$ represents an infinite set of commuting integrals of motion. The idea is to solve their simultaneous diagonalisation problem, much in common to the QISM for the integrable theory. Remarkably, this is equivalent to solving the quantum KdV problem, since at the classical limit the field $T(u) = -\frac{c}{6} U(u)$ with $U(u+2\pi) = U(u)$ reduces the commutators of $T(u)$ to $\{U(u), U(v)\} = 2(U(u) - U(v)) \delta'(u-v) + \delta''(u-v)$, which is the well known Poisson structure of the KdV.

Another practical application of this relationship is to extract the important information about the underlying CFT in the scaling limit of the integrable lattice models. Interestingly, from the finite size correction of the Bethe ansatz solutions, one can determine [28] the CFT characteristics like the central charge and the conformal dimensions. For example one may analyse the finite size effect of the Bethe ansatz solutions of the six-vertex model (with a seam given by κ). Considering the coupling parameter $q = e^{\frac{\kappa}{\nu+1}}$, one obtains from the Bethe solution at the large N limit the expression

$$E_0 = Nf_{\infty} - \frac{1}{N} \frac{\pi}{6} c + O\left(\frac{1}{N^2}\right)$$

for the ground state energy and

$$E_m - E_0 = \frac{2\pi}{N} (\Delta + \bar{\Delta}) + O\left(\frac{1}{N^2}\right)$$

$$P_m - P_0 = \frac{2\pi}{N} (\Delta - \bar{\Delta}) + O\left(\frac{1}{N^2}\right)$$

for the excited states. Here $\Delta, \bar{\Delta}$ are conformal weights of unitary minimal models and $c = 1 - \frac{6\kappa^2}{\nu(\nu+1)}$, $\nu = 2, 3, \dots$ is the central charge of the corresponding conformal field theory.

7.3. Link polynomial using integrable systems :

A link polynomial is an invariants corresponding to a particular knot or link and is extremely useful for classifying them. Jones polynomial is such an example. There are various ways to construct such polynomials. Interestingly, the Integrable systems provide a systematic highly efficient way of producing such polynomials, which can distinguish between different knots, where even Jones polynomial fails. The main idea is to start with a trigonometric $R(\lambda)$ -matrix solution of the YBE, which in general is a $N^2 \times N^2$ matrix depending on the higher representation of the $SU(2)$ algebra. Then the task is to find the corresponding braid group representation by taking the $\lambda \rightarrow \infty$ limit. Defining now the *Markov trace* in a particular way one can construct a series of link polynomials for different cases of $N = 2, 3, \dots$. Higher the N richer is the contents of the polynomial. For example, using $N = 2$ one gets the same polynomial for the Birman's

two closed braids while $N = 3$ by the above method generates two distinct polynomials for these braids [25].

8. Conclusion without conclusion

Basic notions of the quantum integrable systems are explained focusing on various aspects and achievements of this theory. The deep interrelations of this subject with many other fields of physical and mathematical sciences are mentioned. However it is difficult yet to draw any conclusion at this stage, since we expect to hear many more surprises in this evergrowing field. The recent Seiberg-Witten theory might be one of them. The influences of this theory in explaining high T_c -superconductivity [26], reaction-diffusion processes [27] etc. are being felt. We expect also to have breakthrough of quantum integrability in genuine higher dimensions. Therefore let us leave this conclusion without concluding and keep this task for the future.

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